

THE COHOMOLOGY OF CERTAIN HOPF ALGEBRAS ASSOCIATED WITH p -GROUPS

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ABSTRACT. We study the cohomology $H^*(A) = \text{Ext}_A^*(k, k)$ of a locally finite, connected, cocommutative Hopf algebra A over $k = \mathbb{F}_p$. Specifically, we are interested in those algebras A for which $H^*(A)$ is generated as an algebra by $H^1(A)$ and $H^2(A)$. We shall call such algebras *semi-Koszul*. Given a central extension of Hopf algebras $F \rightarrow A \rightarrow B$ with F monogenic and B semi-Koszul, we use the Cartan-Eilenberg spectral sequence and algebraic Steenrod operations to determine conditions for A to be semi-Koszul. Special attention is given to the case in which A is the restricted universal enveloping algebra of the Lie algebra obtained from the mod- p lower central series of a p -group. We show that the algebras arising in this way from extensions by $\mathbb{Z}/(p)$ of an abelian p -group are semi-Koszul. Explicit calculations are carried out for algebras arising from rank 2 p -groups, and it is shown that these are all semi-Koszul for $p \geq 5$.

1. INTRODUCTION

A *quadratic* algebra over a field k is an associative algebra of the form $A = T(V)/(R)$, where (R) is the two-sided ideal of relations generated by some $R \subset V \otimes V$, and V is a locally finite graded vector space. In [13], Priddy defines a quadratic algebra A to be a *Koszul algebra* if $H^*(A) = \text{Ext}_A^*(k, k)$ is generated as an algebra by $\text{Ext}_A^1(k, k)$. Basic examples include tensor, polynomial, and exterior algebras.

When A is Koszul, Priddy constructs projective resolutions over A which are “small” subcomplexes of the bar resolution. He also shows that if $H^*(A)$ is itself Koszul, then $A \simeq H^*(H^*(A))$ (Koszul duality). Equivalent definitions of Koszulity include:

- (1) A certain complex associated to A , the *Koszul complex*, is acyclic [10, Theorem 1.2].
- (2) There exists a graded projective resolution

$$\cdots \rightarrow P^2 \rightarrow P^1 \rightarrow P^0 \rightarrow k \rightarrow 0$$

of k such that P^i is generated by its component of degree i , $P^i = A \cdot P_i^i$ [2, Definition 1.2.1]. Equivalently, $\text{Ext}_A^{i,j}(k, k) = 0$ if $i \neq j$.

Now assume $A = T(V)/(R)$ is a more general algebra which is not necessarily quadratic, i.e. $R \subset \sum_{i \geq 1} V^{\otimes i}$. We are interested in the case when $H^*(A)$ is

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generated as an algebra by $H^1(A)$ and $H^2(A)$. Following Priddy, we shall call such algebras *semi-Koszul*. The quintessential example is the truncated polynomial algebra $\mathbb{F}_p[x]/(x^p)$, whose cohomology is generated by a one-dimensional exterior class z and a two-dimensional polynomial class u . In this example, $u = -\widetilde{\beta\mathcal{P}}^0(z)$. In general, it will often be the case that the generators from H^2 are related to those of H^1 by some algebraic Bockstein operation or Massey product. Since we allow A to not be quadratic, none of the results cited above for Koszul algebras necessarily hold for semi-Koszul algebras. It is unknown at this time whether there are any interesting relationships between A and $H^*(H^*A)$, or anything like a semi-Koszul resolution.

We will focus on locally finite, connected, cocommutative Hopf algebras over $k = \mathbb{F}_p$. Consequently, we will be able to use algebraic Steenrod operations. In Section 3, we analyze extensions of Hopf algebras $F \rightarrow A \rightarrow B$, where $F = \mathbb{F}_p[x]/(x^p)$ is central in A . There is a first quadrant spectral sequence converging to $H^*(A)$ with $E_2^{*,*} \simeq H^*(B) \otimes H^*(F)$. The behavior of this spectral sequence is determined by the differential $d_2(1 \otimes z) = \mu \in H^2(B)$, where $z \in H^1(F)$. We prove in Theorem 3.3 that A is semi-Koszul provided that B is semi-Koszul, $\widetilde{\beta\mathcal{P}}^0(\mu) = 0$ and the annihilator ideal of μ is either zero or generated in dimension one.

Section 4 delves into Hopf algebras arising from p -groups. For a finite p -group G , the mod- p lower central series filtration $\{\Gamma^i G\}$ has as associated graded object a restricted Lie algebra $\mathcal{L}G$. The universal restricted enveloping algebra $\mathcal{V}\mathcal{L}G$ is isomorphic to the associated graded algebra $E^0(\mathbb{F}_p G)$ of the group ring $\mathbb{F}_p G$ with respect to the augmentation ideal filtration [15]. Working with Lie algebras is helpful because many of the algebraic Steenrod operations vanish there, in particular $\widetilde{\mathcal{P}}^0$ [14]. In Section 4.2, we explore central extensions of p -groups $H \rightarrow G \rightarrow Q$, where $H \simeq \mathbb{Z}/(p)$. Using Theorem 3.3, we show that $\mathcal{V}\mathcal{L}G$ is semi-Koszul if Q is abelian (Theorem 4.15). In Section 4.3, we prove a corresponding statement for Hopf algebras which do not necessarily arise from p -groups. Given a central extension of Hopf algebras $F \rightarrow A \rightarrow B$ with $F = \mathbb{F}_p[x]/(x^p)$, we show that $E^0 A$ is semi-Koszul provided that B is commutative.

A rank two p -group is a finite p -group whose largest elementary abelian subgroup has rank two. In Section 5 we show that $\mathcal{V}\mathcal{L}G$ is semi-Koszul for all rank two p -groups G with $p \geq 5$. One would hope to show that this holds for any finite p -group, but such is not the case. A counterexample is given in 5.3.1.

Section 2 introduces notation and covers the necessary algebraic preliminaries such as algebraic Steenrod operations and the Cartan-Eilenberg spectral sequence associated with a central extension of algebras. We relegate to the Appendix definitions of the cobar complex $C^*(A)$, Massey products in $H^*(A)$ and computations in the cohomology of $\mathbb{F}_p[x]/(x^{p^n})$.

2. ALGEBRAIC PRELIMINARIES

In this section we introduce notation and conventions. For more details on the cohomology of algebras, the reader is referred to Adams [1, Chapter 2]. We also summarize algebraic Steenrod operations, the Cartan-Eilenberg spectral sequence, and Lie algebras.

Let $k = \mathbb{F}_p$ be the field of p elements. By an *algebra* A we mean a non-negatively graded, associative k -algebra with product $\mu : A \otimes A \rightarrow A$, unit $\eta : k \rightarrow A$

and augmentation $\epsilon : A \rightarrow k$. We assume A is locally finite, that is, each A_n is finite dimensional over k . We also assume A is *connected*, i.e. $A_0 = k$. Let $I = I(A) = \ker \epsilon$ denote the augmentation ideal.

Tensor products are taken over k , unless otherwise noted. The tensor product of two algebras A and B is defined by $(A \otimes B)_n = \sum_{i+j=n} A_i \otimes B_j$, with multiplication $(a \otimes b)(x \otimes y) = (-1)^{\deg b \deg x} ax \otimes by$. If the algebras in question are bigraded, we define $(A \otimes B)_{n,m} = \sum_{\substack{i+s=n \\ j+t=m}} A_{i,j} \otimes B_{s,t}$, and the total degree is used in the multiplication formula. For $x \in A_{s,t}$, x has *homological* degree (or dimension) s , *internal* degree t and *total* degree $s+t$. By the *commutativity* of A we shall mean *graded-commutativity*, also called *anticommutativity*: $ab = (-1)^{\deg a \deg b} ba$.

A *Hopf algebra* is an algebra with *coproduct*, or *diagonal*, $\psi : A \rightarrow A \otimes A$ which is an algebra map. We shall assume that ψ is (graded) coassociative and cocommutative. Since the algebra A is assumed to be connected, it has an antipode $\sigma : A \rightarrow A$, but we shall not make use of it. See [12] for more details on Hopf algebras and their structure.

Let $\text{Ext}_A^i(-, k)$ denote the i -th right derived functor of $\text{Hom}_A(-, k)$ in the category of graded left A -modules. The *cohomology* $H^*(A)$ of A is defined to be $\text{Ext}_A^*(k, k)$, where k is given an A -module structure by the augmentation map ϵ . The Yoneda product makes this a graded algebra [1]. We recall some basic facts about the cohomology of algebras.

- (1) Let A, B be locally finite algebras. Then $H^*(A \otimes B) \simeq H^*(A) \otimes H^*(B)$.
- (2) If A is a connected cocommutative Hopf algebra with unit, then $H^*(A)$ is anticommutative.

We introduce the following definitions.

Definition 2.1.

- (1) A graded algebra C is called *bigenenerated* if it is generated as an algebra by elements of homological degree 1 and 2.
- (2) An algebra A is *semi-Koszul* if $C = H^*(A)$ is bigenerated.

We shall make use of a reindexed form of *algebraic Steenrod operations*. In [11, Theorem 11.8], May defines operations $\mathcal{P}^i : H^{s,t}(A) \rightarrow H^{s+(2i-t)(p-1), pt}(A)$ for $p \geq 3$. These operations are zero if $2i < t$ or $2i > s+t$. We would like to reindex these so that \mathcal{P}^i raises homological degree by $2i(p-1)$ and is trivial for $i < 0$, $2i > s$. (May denotes these reindexed operations by $\tilde{\mathcal{P}}^i$.) This regrading has the effect of making $\tilde{\mathcal{P}}^0$ the first possibly non-trivial operation on $H^*(A)$. It also eliminates all operations on $H^{s,t}(A)$ for $p > 2$ when t is odd. This will not be of concern to us, as most of our algebras will be concentrated in even degrees.

Theorem 2.2. *Let A be a cocommutative Hopf algebra over \mathbb{F}_p . There exist natural homomorphisms*

- (1) $\tilde{Sq}^i : H^{s,t}(A) \rightarrow H^{s+i, 2t}(A)$ for $p = 2$,
- (2) $\tilde{\mathcal{P}}^i : H^{s, 2t}(A) \rightarrow H^{s+2i(p-1), 2pt}(A)$ and $\tilde{\beta\mathcal{P}}^i : H^{s, 2t}(A) \rightarrow H^{s+2i(p-1)+1, 2pt}(A)$ for $p \geq 3$

with the following properties.

- (1) $\widetilde{Sq}^i = 0$ if $i < 0$ or $i > s$,
 $\widetilde{\mathcal{P}}^i = 0$ if $i < 0$ or $2i > s$,
 $\widetilde{\beta\mathcal{P}}^i = 0$ if $i < 0$ or $2i \geq s$;
- (2) $\widetilde{Sq}^i(x) = x^2$ if $i = s$,
 $\widetilde{\mathcal{P}}^i(x) = x^p$ if $2i = s$.
- (3) Cartan formulae
 $\widetilde{Sq}^i(xy) = \sum_{j=0}^i \widetilde{Sq}^j(x) \widetilde{Sq}^{i-j}(y)$,
 $\widetilde{\mathcal{P}}^i(xy) = \sum_{j=0}^i \widetilde{\mathcal{P}}^j(x) \widetilde{\mathcal{P}}^{i-j}(y)$,
 $\widetilde{\beta\mathcal{P}}^i(xy) = \sum_{j=0}^i (\widetilde{\beta\mathcal{P}}^j(x) \widetilde{\mathcal{P}}^{i-j}(y) + \widetilde{\mathcal{P}}^j(x) \widetilde{\beta\mathcal{P}}^{i-j}(y))$;
- (4) Adem relations
 - (a) if $a < 2b$,
 $\widetilde{Sq}^a \widetilde{Sq}^b = \sum_i \binom{b-i-1}{a-2i} \widetilde{Sq}^{a+b-i} \widetilde{Sq}^i$,
 - (b) if $a < pb$,
 $\widetilde{\beta\epsilon\mathcal{P}}^a \widetilde{\mathcal{P}}^b = \sum_i (-1)^{a+i} \binom{(b-i)(p-1)-1}{a-pi} \widetilde{\beta\epsilon\mathcal{P}}^{a+b-i} \widetilde{\mathcal{P}}^i$,
 - (c) if $a \leq pb$,
 $\widetilde{\beta\epsilon\mathcal{P}}^a \widetilde{\mathcal{P}}^b = (1-\epsilon) \sum_i (-1)^{a+i} \binom{(b-i)(p-1)-1}{a-pi} \widetilde{\beta\mathcal{P}}^{a+b-i} \widetilde{\mathcal{P}}^i$
 $- \sum_i (-1)^{a+i} \binom{(b-i)(p-1)-1}{a-pi-1} \widetilde{\beta\epsilon\mathcal{P}}^{a+b-i} \widetilde{\beta\mathcal{P}}^i$,

where $\epsilon = 0$ or 1 , the binomial coefficients are taken modulo p and, by abuse of notation, $\widetilde{\beta^0\mathcal{P}}^s = \widetilde{\mathcal{P}}^s$ and $\widetilde{\beta^1\mathcal{P}}^s = \widetilde{\beta\mathcal{P}}^s$.

Remark. One can actually define operations $\widetilde{\mathcal{P}}^{i/2}$, $\widetilde{\beta\mathcal{P}}^{i/2}$ on elements of odd internal degree such that the Cartan formulae and Adem relations still make sense. See [16, Appendix 1.5] for details.

A *decreasing filtration* of A is a sequence of k -modules $A = F^0 A \supset F^1 A \supset F^2 A \supset \dots$ such that $F^s A \cdot F^t A \subset F^{s+t} A$. Let ρ_s denote the natural map $F^s A \twoheadrightarrow F^{s/s+1} A = F^s A / F^{s+1} A$. We will sometimes suppress the A when it is clear from context. We will say that elements in F^{s+i} are of *lower* filtration level than elements in F^s , even though the indices are greater. Let the *filtration degree* of an element $x \in A$ be the largest i such that $x \in F^i A$ (if such an i exists).

Let $F \subset A$ be a subalgebra. We say F is *normal* in A if $I(F) \cdot A = A \cdot I(F)$. In this case, there is a well-defined quotient $B = A / A \cdot I(F)$ denoted by $A // F$. In the case that F is a sub-Hopf algebra, B is also a Hopf algebra. We say F is *central* in A if each element of F anticommutes with each element of A . By an *extension* or *exact sequence* $F \rightarrow A \rightarrow B$ of algebras we shall mean that F is normal in A and $B \simeq A // F$.

Theorem 2.3. *Let $F \xrightarrow{i} A \xrightarrow{\pi} B$ be a central extension of graded, connected algebras. There is a first quadrant spectral sequence converging to $H^*(A)$ with*

- (1) $E_2^{s,t} \simeq H^s(B) \otimes H^t(F)$.
- (2) $E_\infty^{s,t} \simeq F^s H^{s+t}(A) / F^{s+1} H^{s+t}(A)$ where $\{F^s\}$ is a decreasing filtration of $H^*(A)$.
- (3) The ring structure of $H^s(B) \otimes H^t(F)$ is defined by
 $(x \otimes y)(z \otimes w) = (-1)^{(t+t')(s+s')} xz \otimes yw$, where $y \in H^{t,t'}(F)$, $z \in H^{s,s'}(B)$.

- (4) $d_r : E_r^{s,t} \rightarrow E_r^{s+r,t-r+1}$ is a graded derivation, meaning $d_r(ab) = d_r(a) \cdot b + (-1)^{\deg a} a \cdot d_r(b)$.
- (5) The map $H^s(B) \simeq E_2^{s,0} \rightarrow E_\infty^{s,0} \simeq F^s H^s(A) \hookrightarrow H^s(A)$ corresponds to the induced map $\pi^* : H^*(B) \rightarrow H^*(A)$.
The map $H^t(A) = F^0 H^t(A) \simeq E_\infty^{0,t} \hookrightarrow E_2^{0,t} \simeq H^t(F)$ corresponds to the induced map $i^* : H^*(A) \rightarrow H^*(F)$.
- (6) Kudo transgression theorem. Assume, in addition, that $F \xrightarrow{i} A \xrightarrow{\pi} B$ is an extension of Hopf algebras over \mathbb{F}_p . The transgression $d_r : E_r^{0,r-1} \rightarrow E_r^{r,0}$ anticommutes with Steenrod operations: if $x \in E_r^{0,r-1}$ and θ is a Steenrod operation of homological degree i , then $\theta(x)$ survives until $E_{r+i}^{0,r+i-1}$ and $d_{r+i}(\theta(x)) = (-1)^i \theta(d_r(x))$.

Since the algebras F , A and B are graded, the spectral sequence is actually trigraded, with the third grading being the internal degree. We have suppressed this third grading, as it is preserved by the differentials d_r . This spectral sequence was used by Adams [1, 2.3.1]. Ravenel [16, A1.3.14] calls it the *Cartan-Eilenberg* spectral sequence. For (6) see [17, Proposition 2.3].

The Lie algebras we shall work with are all graded over either \mathbb{Z} or \mathbb{F}_p . A *restricted* Lie algebra is a Lie algebra over \mathbb{F}_p with restriction ξ .

For L a restricted Lie algebra, let $\text{Tens}(L)$ be the tensor algebra on L , and let $J \subset \text{Tens}(L)$ be the two-sided ideal generated by elements of the form $x \otimes y - (-1)^{\deg x \deg y} y \otimes x - [x, y]$ and $x^{\otimes p} - \xi(x)$ for $x, y \in L$.

Definition 2.4. The *universal restricted enveloping algebra* of L is defined to be $\mathcal{V}L = \text{Tens}(L)/J$.

We can put a Hopf algebra structure on $\mathcal{V}L$ by setting $\psi(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$ for $x \in L$. The cohomology of a Lie algebra L is defined to be $H^*(\mathcal{V}L)$.

Theorem 2.5 ([14, 5.3], [11, Theorem 8.5]). *Let L be a Lie algebra over \mathbb{F}_p . Then the Steenrod operations $\widetilde{\mathcal{P}}^0$ and \widetilde{Sq}^0 are identically zero on $H^*(\mathcal{V}L)$.*

3. AN EXTENSION THEOREM

Let A be a cocommutative Hopf algebra over \mathbb{F}_p . In this section, we will find sufficient conditions for $H^*(A)$ to be bigenerated. Let $z \in A$ be a central primitive element of height p . This means z anticommutes with every element of A , $z^p = 0$ and $z^i \neq 0$ if $i < p$. Let $F = \mathbb{F}_p[z]/(z^p)$ be the central sub-Hopf algebra of A generated by z . Let $B = A//F$. We have the central extension of Hopf algebras

$$(3.1) \quad F \longrightarrow A \longrightarrow B$$

with corresponding spectral sequence

$$(3.2) \quad E_2^{s,t} \simeq H^s(B) \otimes H^t(F) \Rightarrow H^{s+t}(A).$$

Let n be the internal degree of $z \in F$. Note that when p is odd, n must be even. We recall that the cohomology of the fiber F is

$$\begin{aligned} H^*(F) &\simeq \Lambda[\zeta] \otimes \mathbb{F}_p[\epsilon], & \zeta &\in H^{1,n}, \epsilon \in H^{2,np}, & \text{with } \widetilde{\beta\mathcal{P}}^0(\zeta) &= -\epsilon & p \text{ odd,} \\ H^*(F) &\simeq \mathbb{F}_2[\zeta], & \zeta &\in H^{1,n}, & \text{with } \widetilde{Sq}^1(\zeta) &= \zeta^2 & p = 2. \end{aligned}$$

See Theorem A.9 for more details on the Steenrod operations.

Theorem 3.3 (Main Theorem). *Let F, A, B be as in (3.1). In the spectral sequence (3.2), let $d_2(\zeta) = \mu \in H^{2,n}(B)$. Assume*

- (1) $\widetilde{\beta\mathcal{P}}^0(\mu) = 0$ or $\widetilde{Sq}^1\mu = 0$.
- (2) *The annihilator ideal of μ in $H^*(B)$ is either zero or is generated in dimension one.*
- (3) *B is semi-Koszul.*

Then A is semi-Koszul.

Proof. By the transgression theorem 2.3(6),

$$\begin{aligned} d_3(\epsilon) &= -\widetilde{\beta\mathcal{P}}^0(\mu), \quad p \text{ odd}, \\ d_3(\zeta^2) &= \widetilde{Sq}^1(\mu), \quad p = 2. \end{aligned}$$

If $\widetilde{\beta\mathcal{P}}^0(\mu) = 0$ or $\widetilde{Sq}^1(\mu) = 0$, then

$$\begin{aligned} d_{2p+1}(\epsilon^p) &= \widetilde{\mathcal{P}}^1(d_3(\epsilon)) = 0 \quad \text{or} \\ d_5(\zeta^4) &= \widetilde{Sq}^2(d_3(\zeta^2)) = 0. \end{aligned}$$

Likewise for all other powers of ϵ or ζ . Thus in this case, the spectral sequence (3.2) collapses at E_3 .

In order for $H^*(A)$ to be bigenerated, we need E_∞ to be bigenerated. Assume $\widetilde{\beta\mathcal{P}}^0(\mu) = 0$ or $\widetilde{Sq}^1(\mu) = 0$. If μ is a zero divisor in $H^*(B)$, we may run into the following problem. Let $\eta \in H^i(B)$ be in the annihilator ideal of μ . Then

$$d_2(\eta \otimes \zeta) = d_2(\eta) \otimes \zeta + (-1)^i \eta \otimes d_2(\zeta) = \pm \eta \otimes \mu = 0.$$

Nothing can hit $\eta \otimes \zeta \in E_r^{i,1}$ because $d_r \equiv 0$ for $r \geq 3$, and $d_2 \equiv 0$ from the $E_2^{*,2}$ line. Thus $\eta \otimes \zeta$ is a non-bounding permanent cycle of dimension $i + 1$. If $i = 1$, then $\eta \otimes \zeta \in E_\infty^{1,1}$ is a two-dimensional generator of E_∞ . If $i > 1$ on the other hand, $\eta \otimes \zeta$ is a possibly indecomposable element in $E_\infty^{i,1}$ of homological dimension larger than 2. (It will be indecomposable if, for example, η is indecomposable in $H^*(B)$.) Consequently E_∞ —and thus $H^*(A)$ —is not bigenerated.

Assume then that the annihilator ideal of μ in $H^*(B)$ is either zero or is generated multiplicatively by elements of dimension one. Assume also that $H^*(B)$ is bigenerated. Then E_∞ has the following possible set S of multiplicative generators:

- (1) $1 \otimes \epsilon$ for p odd, $1 \otimes \zeta^2$ for $p = 2$;
- (2) elements of the form $\eta \otimes \zeta$, where $\eta \in H^1(B)$ and $\eta\mu = 0$;
- (3) elements $\sigma \otimes 1$, $\nu \otimes 1$, where $\sigma \in H^1(B)$ and $\nu \in H^2(B)$;
- (4) $1 \otimes \zeta$ (if $d_2 \equiv 0$).

All these elements have homological degree less than or equal to two, and thus E_∞ is bigenerated. Recall that $H^*(A)$ has a decreasing filtration $F^*H^*(A)$, and that we denote the natural projection $F^sH^n(A) \rightarrow F^{s/s+1}H^n(A) \simeq E_\infty^{s,n-s}$ by ρ_s . Choose a set of representatives $T \subset H^*(A)$ for each of the types of generators in S :

- (1) $e \in F^0H^2(A)$;
- (2) $x \in F^1H^2(A)$;
- (3) $\alpha \in F^1H^1(A)$, $v \in F^2H^2(A)$;
- (4) $z \in F^0H^1(A)$.

We claim that $H^*(A)$ is generated as an algebra by T , from which it follows that A is semi-Koszul. Let $S' = S \cup \{1 \otimes 1\}$, $T' = T \cup \{1\}$. The proof will proceed by reverse induction on filtration degree s in $H^n(A)$. Let $y \in F^n H^n(A)$. Then $y \in H^n(B)$, which is bigenerated. Now let $y \in F^s H^n(A)$, where $s < n$, and assume $F^i H^n(A)$ is bigenerated for $i > s$. Projecting y to $F^{s/s+1} H^n(A) \simeq E_\infty^{s,n-s}$, $\rho_s(y)$ can be written as a sum of products $f_i \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n}$, where $f_i \in k$ and $\lambda_{i_j} \in S'$. So y is a sum of products $f_i \tilde{\lambda}_{i_1} \cdots \tilde{\lambda}_{i_n}$ and a term γ , where $\tilde{\lambda}_{i_j} \in T'$ are lifts of λ_{i_j} under ρ , and $\gamma \in F^{s+1} H^n(A)$. By induction, γ is also a linear combination of products of elements from T' . Thus y is of the desired form. \square

4. HOPF ALGEBRAS ARISING FROM p -GROUPS

The mod- p lower central series of a group G gives rise to a restricted Lie algebra L , whose universal restricted enveloping algebra VL is a Hopf algebra. Starting with a central extension of p -groups $H \rightarrow G \rightarrow Q$, we take enveloping algebras of the associated Lie algebras and get a central extension of Hopf algebras $F \rightarrow A \rightarrow B$. In Section 4.2 we show that A is semi-Koszul if Q is abelian and $H \simeq \mathbb{Z}/(p)$. In Section 4.3, we generalize to extensions of Hopf algebras $F \rightarrow A \rightarrow B$ which do not necessarily arise from extensions of p -groups. We show that $E^0 A$ is semi-Koszul if B is commutative.

4.1. Definitions. Let G be a group, and let $G = F^1 G \supseteq F^2 G \supseteq \cdots$ be a descending sequence of subgroups (also called a filtration) with $F^i G \supseteq F^{i+1} G$, $F^i G / F^{i+1} G$ abelian. Let $\text{gr}^F G = \bigoplus_i F^i G / F^{i+1} G$ be the *associated graded* object, and let $\text{gr}_i^F G = F^i G / F^{i+1} G$. When the filtration is understood, we shall simply write $\text{gr} G$ and $\text{gr}_i G$. Recall the natural surjection $\rho_i : F^i G \rightarrow \text{gr}_i G$.

For H, K subgroups of G , let (H, K) be the subgroup generated by all commutators of the form $(h, k) = h^{-1} k^{-1} h k$ for $h \in H, k \in K$. Let H^n be the subgroup generated by h^n for $h \in H$. An N -sequence is a normal series $G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots$ such that $(G_i, G_j) \subset G_{i+j}$ [9]. The successive quotients $\text{gr}_i G = G_i / G_{i+1}$ of an N -sequence are abelian, since $(G_i, G_i) \subset G_{2i} \subset G_{i+1}$. The associated graded abelian group $\text{gr} G = \bigoplus_n G_n / G_{n+1}$ has the structure of a Lie algebra over \mathbb{Z} , with bracket $[\rho_r x, \rho_s y] = \rho_{r+s}((x, y))$. A p -restricted N -sequence is an N -sequence $\{G_n\}$ such that $G_n^p \subset G_{pn}$. The associated graded group $\text{gr} G$ is then a vector space over \mathbb{F}_p . We can define a restriction $\xi(\rho_r(x)) = \rho_{pr}(x^p)$ on $\text{gr} G$ which makes it into a restricted Lie algebra over \mathbb{F}_p .

The fastest descending p -restricted N -sequence of G is the *mod- p lower central series* $\{\Gamma^i G\}$ (see [9], [5]):

$$(4.1) \quad \Gamma^{2n-1} G = \Gamma^{2n} G = \langle (\gamma_1, \dots, \gamma_r)^{p^u} \text{ such that } r \cdot p^u \geq n \rangle,$$

where $(\gamma_1, \dots, \gamma_r) = (\gamma_1, (\dots (\gamma_{r-2}, (\gamma_{r-1}, \gamma_r)) \dots))$. The associated object $\text{gr}^\Gamma G$ is thus a restricted Lie algebra over \mathbb{F}_p . Let $\mathcal{L}(G) = \mathcal{L}G = \text{gr}^\Gamma G$. It can easily be checked that \mathcal{L} is a functor from the category of groups to the category of restricted Lie algebras over \mathbb{F}_p .

The definition we have given here is slightly different from that of [15] or [5]. Our Γ^{2n} equals their Γ^n . We have doubled the filtration degree in order not to have to worry about signs for p odd.

Let kG be the group ring of G over k , with augmentation $\epsilon(g) = 1$ for $g \in G$. A diagonal $\psi : kG \rightarrow kG \otimes kG$ is induced by the map $g \rightarrow g \otimes g$, giving

kG the structure of a cocommutative Hopf algebra over k . Let $I = \ker \epsilon$ be the augmentation ideal of kG . Filter kG as follows: $F^{2n-1} = F^{2n} = I^n$ for $n > 0$. The associated graded algebra $E^0(kG) = \text{gr}^F kG$ has the structure of a restricted Lie algebra over k inherited from the multiplication in kG . Specifically, $[\rho_r x, \rho_s y] = \rho_{r+s}(xy - yx)$, $\xi(\rho_r x) = \rho_{rp}(x^p)$. Since ψ respects the filtration on kG , it induces a cocommutative diagonal on $E^0(kG)$. It is clear that $E^0(kG)$ is generated as an algebra by $E_2^0(kG) = I/I^2$. These elements are all primitive by reason of dimension. Thus $E^0(kG)$ is primitively generated, and by a theorem of Milnor-Moore [12, Theorem 6.11], $E^0(kG)$ is isomorphic to the restricted universal enveloping algebra of its restricted Lie subalgebra $L = E_2^0 kG$.

The map $\psi : \mathcal{L}G \rightarrow E^0(kG)$ given by $\rho_r(x) \mapsto \rho_r(x-1)$ is a restricted Lie algebra homomorphism. In fact, the image of ψ lies in the primitive elements $\mathcal{P}(E^0(kG))$ of $E^0(kG)$.

Theorem 4.2 ([15, Theorem 1.4]). *The induced map*

$$\mathcal{L}G \longrightarrow \mathcal{P}(E^0(kG))$$

is an isomorphism of restricted Lie algebras.

Therefore,

$$(4.3) \quad \mathcal{V}(\mathcal{L}G) \simeq \mathcal{V}(\mathcal{P}(E^0(kG))) \simeq E^0(kG)$$

as Hopf algebras over k .

Let $\mathcal{V}\mathcal{L}G$ denote $\mathcal{V}(\text{gr}^F G)$. Studying the cohomology $H^*(E^0(kG))$ over k is thus equivalent to studying $H^*(\mathcal{V}\mathcal{L}G)$. The advantage of working with cohomology of Lie algebras is that the Steenrod operations $\widetilde{Sq}^0 \equiv 0$, $\widetilde{P}^0 \equiv 0$ there ([14], [11, Theorem 8.5]), which greatly simplifies the situation.

We shall need the following result.

Lemma 4.4. *Let G be a finite abelian p -group. Then $\mathcal{V}\mathcal{L}G$ is a semi-Koszul algebra.*

Proof. Let $G = \mathbb{Z}/(p^n)$. It is clear that

$$\mathcal{V}\mathcal{L}G \simeq E^0(\mathbb{F}_p G) \simeq \mathbb{F}_p[x]/(x^{p^n}).$$

For a general finite p -group $G = \mathbb{Z}/(p^{n_1}) \times \cdots \times \mathbb{Z}/(p^{n_k})$, then

$$\mathcal{V}\mathcal{L}G \simeq \mathbb{F}_p[x_1, \dots, x_k]/(x_i^{p^{n_i}}).$$

Thus

$$H^*(\mathcal{V}\mathcal{L}G) \simeq \Lambda[z_1, \dots, z_k] \otimes \mathbb{F}_p[e_1, \dots, e_k],$$

where $z_i \in H^{1,2}$, $e_i \in H^{2,2p^{n_i}}$. If $p = 2$ and some $n_i = 1$, replace $\Lambda[z_i] \otimes \mathbb{F}_p[e_i]$ by $\mathbb{F}_2[z_i]$. \square

4.2. Extensions of p -groups. Let G be a non-trivial finite p -group, and let $k = \mathbb{F}_p$. There is an element x of order p in the center of G . Let $H \simeq \mathbb{Z}/(p)$ be the subgroup generated by x . We have the central extension

$$(4.5) \quad H \xrightarrow{f} G \xrightarrow{g} Q = G/H.$$

Our plan is to apply the \mathcal{L} functor, and then the \mathcal{V} functor, to (4.5) in the hope of obtaining an exact sequence of Hopf algebras of the form (3.1).

First, note that $g(\Gamma^n G) = \Gamma^n Q$. Refilter H by $F^n H = f^{-1}(\Gamma^n G)$. According to Theorem 2.4 of [9], the following is an exact sequence of graded Lie algebras over \mathbb{Z} :

$$(4.6) \quad \text{gr.}^F H \xrightarrow{\bar{f}} \text{gr.} G \xrightarrow{\bar{g}} \text{gr.} Q.$$

Now $\text{gr.}^F H$ is generated by $\bar{x} \in F^{2n/2n+1} H$, where $2n$ is the filtration degree of $f(x)$ in G . Since $\mathcal{L}H$ is generated by $\bar{x} \in \Gamma^{2/3} H$, $\text{gr.}^F H \simeq \mathcal{L}H$ as ungraded Lie algebras. The sequence (4.6) is actually a sequence of graded restricted Lie algebras and—if we regrade $\mathcal{L}H$ appropriately—is isomorphic to

$$(4.7) \quad \mathcal{L}H \longrightarrow \mathcal{L}G \longrightarrow \mathcal{L}Q.$$

Now let

$$(4.8) \quad L_1 \xrightarrow{i} L_2 \xrightarrow{\pi} L_3$$

be an exact sequence of graded restricted Lie algebras over \mathbb{F}_p . Recall that $\mathcal{V}L_i$ is defined to be the quotient of $\text{Tens}(L_i)$ by some specific ideal J_i . It is clear that $\pi(J_2) = J_3$, and that $i^{-1}(J_2) = J_1$. Therefore,

$$(4.9) \quad \text{Tens}(L_1)/J_1 \longrightarrow \text{Tens}(L_2)/J_2 \longrightarrow \text{Tens}(L_3)/J_3$$

is an exact sequence of Hopf algebras over k .

We have just proven

Proposition 4.10. *Let H, G , and Q be as in (4.5). Regrade $\mathcal{V}LH$ as in (4.7). Then the following sequence of Hopf algebras is exact:*

$$(4.11) \quad \mathcal{V}LH \longrightarrow \mathcal{V}LG \longrightarrow \mathcal{V}LQ.$$

Since $\mathcal{V}LH \simeq \mathbb{F}_p[x]/(x^p)$, the central extension (4.11) is of the form $F \rightarrow A \rightarrow B$, as in (3.1). Associated to this extension is the Cartan-Eilenberg spectral sequence (3.2):

$$(4.12) \quad \begin{aligned} E_2 &\simeq H^*(\mathcal{V}LQ) \otimes \Lambda[z] \otimes \mathbb{F}_p[e] \Rightarrow H^*(\mathcal{V}LG), & p \text{ odd}, \\ E_2 &\simeq H^*(\mathcal{V}LQ) \otimes \mathbb{F}_2[z] \Rightarrow H^*(\mathcal{V}LG), & p = 2. \end{aligned}$$

Let us now examine conditions on the group G which will assure that the assumptions of Theorem 3.3 are satisfied.

Theorem 4.13. *In the spectral sequence (4.12), assume $\mu = d_2(z)$ is a sum of products of one-dimensional elements. That is, assume μ is decomposable. Assume also that conditions (2) and (3) of Theorem 3.3 hold. Then $\mathcal{V}LG$ is semi-Koszul.*

Proof. This result follows from the fact that

$$(4.14) \quad \begin{aligned} \widetilde{\beta\mathcal{P}}^0(\sigma\tau) &= \widetilde{\beta\mathcal{P}}^0(\sigma)\widetilde{\mathcal{P}}^0(\tau) + \widetilde{\mathcal{P}}^0(\sigma)\widetilde{\beta\mathcal{P}}^0(\tau) = 0 \text{ and} \\ \widetilde{Sq}^1(\sigma\tau) &= \widetilde{Sq}^1(\sigma)\widetilde{Sq}^0(\tau) + \widetilde{Sq}^0(\sigma)\widetilde{Sq}^1(\tau) = 0, \end{aligned}$$

since $\widetilde{\mathcal{P}}^0 \equiv 0$ and $\widetilde{Sq}^0 \equiv 0$ on the cohomology of a Lie algebra. Condition (1) of Theorem 3.3 is thus satisfied. \square

Theorem 4.15. *If Q is abelian, then $\mathcal{V}LG$ is semi-Koszul.*

Proof. By Lemma 4.4, $H^*(\mathcal{VL}Q)$ is isomorphic to a tensor product of algebras of the form $\Lambda[z_i] \otimes \mathbb{F}_p[e_i]$ (and $\mathbb{F}_2[z]$ for $p = 2$), so $H^2(\mathcal{VL}Q)$ is spanned by $\{z_i z_j, e_i\}$. Now $\widetilde{\beta\mathcal{P}}^0(z_i z_j) = 0 = \widetilde{Sq}^1(z_i z_j)$ by (4.14) and $\widetilde{\beta\mathcal{P}}^0(e_i) = 0$ by reason of internal degree. Thus condition (1) of Theorem 3.3 is satisfied. Condition (2) is satisfied because the annihilator ideal of $H^2(\mathcal{VL}Q)$ is generated by one-dimensional elements $\{z_i\}$. Finally, condition (3) is satisfied since $H^*(\mathcal{VL}Q)$ is bigenerated. \square

4.3. Generalization to Hopf algebras. The results in §4.2 can be generalized somewhat to Hopf algebras. Let A be a Hopf algebra over \mathbb{F}_p . Assume A contains a central, primitive element x of height p . This means x anticommutes with every element of A , $x^p = 0$, and $x^i \neq 0$ if $i < p$. If p is odd, x is forced to have even internal degree. Let $M = \mathbb{F}_p[x]/(x^p)$ be the central sub-Hopf algebra of A generated by x .

We have the central extension of graded Hopf algebras

$$(4.16) \quad M \xrightarrow{f} A \xrightarrow{g} B.$$

For an augmented algebra C , let $\{F^{2n-1}C = F^{2n}C = (IC)^n\}$ be the modified augmentation ideal filtration, and let $E^0C = \text{gr}^F C$. Since $\epsilon^B(g(y)) = \epsilon^A(y)$, we see that $g(IA) \subset IB$, and more generally, $g(IA^n) \subset IB^n$. Thus g respects the augmentation ideal filtration, and we can form the maps

$$\begin{aligned} \bar{g}_{2n} : F^{2n/2n+1}A &= IA^n/IA^{n+1} \longrightarrow IB^n/IB^{n+1} = F^{2n/2n+1}B, \\ E^0g : E^0A &\longrightarrow E^0B. \end{aligned}$$

Each \bar{g}_{2n} is a map of graded modules, while E^0A and E^0B are *bigraded* algebras (by internal degree and filtration degree). It is easy to check that E^0g is surjective. If $\bar{g}_{2n}(\bar{y}) = 0$, this means $g(y) \in F^{2n+1}B$. Since g is surjective, y can be written as $y = a + b$, where $a \in F^{2n+1}A$ and $b \in \ker g \cap F^{2n}A$. Since $\ker g = \text{im } f$, $b = f(b')$ for some $b' \in M$. Unfortunately, we cannot tell if $b' \in F^{2n}M$. If we refilter M by $\hat{F}^{2n}M = f^{-1}(F^{2n}A)$, we obtain the exact sequence of bigraded algebras

$$\text{gr}^{\hat{F}} M \longrightarrow E^0A \longrightarrow E^0B.$$

In our case, $\text{gr}^{\hat{F}} M \simeq E^0M \simeq M$, except that the generator of $\text{gr}^{\hat{F}} M$ has a filtration degree equal to the filtration degree of $f(x)$ in A . With the understanding that M is regraded as such, let us investigate the extension

$$M \longrightarrow E^0A \longrightarrow E^0B$$

of primitively generated Hopf algebras with associated Cartan-Eilenberg spectral sequence

$$(4.17) \quad E_2^{s,t} \simeq H^s(E^0B) \otimes H^t(M) \Longrightarrow H^{s+t}(E^0A).$$

We recall that a Hopf algebra C is *monogenic* if it is generated by one element x .

Proposition 4.18 ([12]). *A monogenic Hopf algebra over \mathbb{F}_p has the form*

- (1) *An exterior algebra $\Lambda[x] = \mathbb{F}_p[x]/(x^2)$, with $\deg x$ odd for p odd.*
- (2) *A truncated polynomial algebra $A = \mathbb{F}_p[x]/(x^{p^n})$ as in (A.6).*
- (3) *A polynomial algebra $\mathbb{F}_p[x]$ with $\deg x$ even for p odd.*

In all cases, x is primitive by reason of dimension. We have excluded the case $p = 2$, $n = 1$ from (2), as $\mathbb{F}_2[x]/(x^2) \simeq \Lambda[x]$.

Lemma 4.19 ([17, Proposition 2.2]). *For each monogenic Hopf algebra C above,*

- (1) $H^*(C) \simeq \mathbb{F}_p[z]$, with $z \in H^{1, \deg x}$.
- (2) $H^*(C) \simeq \Lambda[z] \otimes \mathbb{F}_p[e]$, with $z \in H^{1, \deg x}$, $e \in H^{2, p^n \deg x}$.
- (3) $H^*(C) \simeq \Lambda[z]$, with $z \in H^{1, \deg x}$.

Theorem 4.20 (Borel, [12]). *A locally finite, connected, primitively generated, commutative Hopf algebra over \mathbb{F}_p is isomorphic to a tensor product of monogenic Hopf algebras.*

Theorem 4.21. *Let M, A, B be as in (4.16). If B is commutative, then $E^0 A$ is semi-Koszul.*

Proof. The fiber M has cohomology generated by $z \in H^1(M)$ and possibly $e \in H^2(M)$. The element e is related to z by the Steenrod operation θ , where θ is one of $\widetilde{\beta\mathcal{P}}^0$ or \widetilde{Sq}^1 . The spectral sequence (4.17) will collapse if $d_3(e) = \theta(d_2(z)) = 0$. Since B is commutative, so is $E^0 B$. By Borel's theorem, $E^0 B \simeq \otimes_i B_i$, where the B_i are monogenic Hopf algebras. Then $H^*(E^0 B) \simeq H^*(B_1) \otimes \cdots \otimes H^*(B_n) \simeq \Lambda[z_i] \otimes \mathbb{F}_p[z'_j] \otimes \mathbb{F}_p[e_k]$ with $z_i, z'_j \in H^1$, $e_k \in H^2$. We can show $\theta \equiv 0$ on $H^2(E^0 B)$ by using the same argument as in Theorem 4.15. \square

Remark. A theorem of Wilkerson [17] states that a *finite-dimensional*, graded, connected, cocommutative Hopf algebra $A \neq k$ always contains a non-trivial central monogenic sub-Hopf algebra C . In the case that C is a truncated polynomial algebra of height p^n , we can always find a central primitive element $x \in A$ of height p . The conditions of Theorem 4.21 then hold.

5. RANK TWO p -GROUPS

In this section we show that the cohomology of the Lie algebra associated with a rank 2 p -group for $p \geq 5$ is bigenerated. A rank 2 p -group is a finite p -group whose largest elementary abelian subgroup has dimension 2. In the case $p \geq 5$, Blackburn [3] has classified these as belonging to one of the following four families: abelian, metacyclic, $C(r)$ and $G(r, e)$ of order p^r . A metacyclic p -group P is an extension of a cyclic group by a cyclic group. A presentation for $C(r)$ is given by

$$C(r) = \langle a, b, c \mid a^p = b^p = c^{p^{r-2}} = 1, (a, b) = c^{p^{r-3}}, c \in \mathcal{Z}(C) \rangle,$$

where $r \geq 3$, $(a, b) = a^{-1}b^{-1}ab$, and $\mathcal{Z}(C)$ is the center of C . A presentation for $G(r, e)$ is given by

$$G(r, e) = \langle f, g, h \mid f^p = g^{p^{r-2}} = h^p = (g, h) = 1, (f, h^{-1}) = g^{ep^{r-3}}, (f, g) = h \rangle,$$

where $r \geq 4$, e is 1 or a quadratic non-residue mod p .

These also define rank 2 p -groups for $p = 3$. We include the families for $p = 3$ in the calculations to follow because they are the source of most of the interesting behavior.

We will be performing explicit calculations in the cobar complex and with Massey products. See the Appendix for a review of these concepts.

5.1. G of type $C(r)$. Since $c^{p^{r-3}}$ is a central element of order p , we have the central extension

$$\mathbb{Z}/(p) \longrightarrow G \longrightarrow \mathbb{Z}/(p) \times \mathbb{Z}/(p) \times \mathbb{Z}/(p^{r-3}).$$

As the base is abelian, Theorem 4.15 tells us that $\mathcal{V}\mathcal{L}G$ is semi-Koszul. We shall now explicitly calculate $H^*(\mathcal{V}\mathcal{L}G)$.

5.1.1. *Case 1: $r \geq 4$.*

$$\begin{aligned} \Gamma^2 &= \langle a, b, c \rangle = G & \Gamma^{2/3} &= \mathbb{F}_p \langle \bar{a}, \bar{b}, \bar{c} \rangle \\ \Gamma^{2p} &= \langle c^p \rangle \simeq \mathbb{Z}/(p^{r-3}) & \Gamma^{2p/2p+1} &= \mathbb{F}_p \langle \overline{c^p} \rangle \\ &\vdots & &\vdots \\ \Gamma^{2p^{r-3}} &= \langle c^{p^{r-3}} \rangle \simeq \mathbb{Z}/(p) & \Gamma^{2p^{r-3}/2p^{r-3}+1} &= \mathbb{F}_p \langle \overline{c^{p^{r-3}}} \rangle \\ & & \Gamma^{i/i+1} &= 0 \text{ for } i \neq 2p^j, j \leq r-3. \end{aligned}$$

Since $a^{-1}b^{-1}ab = c^{p^{r-3}} \in \Gamma^{2p^{r-3}}$ falls below Γ^4 , $\mathcal{L}G$ is an abelian restricted Lie algebra over \mathbb{F}_p . Moreover, $\xi(\bar{a}) = \xi(\bar{b}) = 0$, $\xi(\overline{c^p}) = \overline{c^{p^{t+1}}}$. Let $x = \bar{a}$, $y = \bar{b}$, $z_i = \overline{c^{p^{i-1}}}$ for $1 \leq i \leq r-2$. Then

$$\begin{aligned} \mathcal{V}\mathcal{L}G &= \text{Tens}\{x, y, z_i\} / \left(\begin{array}{l} x^p = y^p = 0, z_i^p = z_{i+1} \\ z_{r-2}^p = 0, x, y, z_i \text{ commute} \end{array} \right) \\ &\simeq \mathbb{F}_p[x, y, z] / (x^p, y^p, z^{p^{r-2}}) \quad \text{as Hopf algebras.} \end{aligned}$$

So $H^*(\mathcal{V}\mathcal{L}C(r)) \simeq \Lambda[x, y, z] \otimes \mathbb{F}_p[u, v, w]$, where $x, y, z \in \text{Ext}^{1,2}$, and $u, v \in \text{Ext}^{2,2p}$ and $w \in \text{Ext}^{2,2p^{r-2}}$. As usual, the one-dimensional classes x, y are related to the two-dimensional classes u, v by the algebraic Bockstein $-\widetilde{\beta}\mathcal{P}^0$, and $w = \langle z \rangle^{p^{r-2}}$. See Appendix A.3 for details.

5.1.2. *Case 2: $r = 3$.* Let $G = C(3) = \langle a, b, c \mid a^p = b^p = c^p = 1, (a, b) = c \in \mathcal{Z}(G) \rangle$. This case is more interesting, as the filtration is different. The filtration degree of c is 4, which makes $\mathcal{L}G$ non-abelian: $[\bar{a}, \bar{b}] = \bar{c} \in \Gamma^{4/5}$. We still have $\xi(\bar{a}) = \xi(\bar{b}) = 0$. Thus $\mathcal{V}\mathcal{L}G \simeq \text{Tens}\{a, b, c\}/J$, where J is the ideal generated by the relations $\{ab - ba = c, ac = ca, bc = cb, a^p = b^p = c^p = 0\}$. The central extension of groups

$$(5.1) \quad \mathbb{Z}/(p) \xrightarrow{z \mapsto c} G \xrightarrow{a, b \mapsto x, y} \mathbb{Z}/(p) \times \mathbb{Z}/(p)$$

induces the central extension of algebras

$$(5.2) \quad \mathbb{F}_p[z]/(z^p) \xrightarrow{z \mapsto c} \mathcal{V}\mathcal{L}G \xrightarrow{a, b \mapsto x, y} \mathbb{F}_p[x, y]/(x^p, y^p)$$

which we abbreviate as $F \longrightarrow A \longrightarrow B$. This sequence will be exact as *graded* algebras if we set $\deg z = 4$. The induced Cartan-Eilenberg spectral sequence has E_2 term

$$(5.3) \quad E_2^{*,*}(C(3)) = H^*(B) \otimes H^*(F) \simeq \Lambda[\sigma, \tau] \otimes \mathbb{F}_p[\mu, \nu] \otimes \Lambda[\zeta] \otimes \mathbb{F}_p[\epsilon],$$

$$\begin{aligned} \sigma, \tau &\in E_2^{1,0,2}, & \mu, \nu &\in E_2^{2,0,2p}, & \mu &= -\widetilde{\beta}\mathcal{P}^0(\sigma) = \langle \sigma \rangle^p, & \nu &= -\widetilde{\beta}\mathcal{P}^0(\tau) = \langle \tau \rangle^p, \\ \zeta &\in E_2^{0,1,4}, & \epsilon &\in E_2^{0,2,4p}, & \epsilon &= -\widetilde{\beta}\mathcal{P}^0(\zeta) = \langle \zeta \rangle^p. \end{aligned}$$

The third superscript is the internal degree. The elements $\sigma \otimes 1$ and $\tau \otimes 1$ in $E_2^{1,0}$ are permanent cycles converging to non-zero classes α and β in $H^{1,2}(A)$. Put a reverse lexicographic order on A : $c > b > a$. Choose the basis for A induced by this ordering: $\mathcal{B} = \{c^i b^j a^k \mid 0 \leq i, j, k \leq p-1\}$. The non-trivial relations are generated by the following:

$$ab = ba + c, \quad ac = ca, \quad bc = cb, \quad a^p = b^p = c^p = 0.$$

Let $\mathcal{B}^* = \{x^* | x \in \mathcal{B}\}$ denote the dual basis, i.e.,

$$\langle x^*, y \rangle = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Identify α, β with the classes $[a^*], [b^*]$ in the cobar complex $C^*(A)$. Now, $\delta[c^*] = [a^*|b^*] \Rightarrow \alpha\beta = 0$ in $E_\infty^{2,0}$. This implies

$$(5.4) \quad d_2(1 \otimes \zeta) = \lambda \cdot \sigma \otimes \tau$$

for some $\lambda \in \mathbb{F}_p^\times$, because that is the last differential which can hit $E_2^{2,0}$. We can ignore the constant λ ; all that matters is that it is non-zero. Since $\sigma \otimes \tau = 1 \otimes \sigma \cdot 1 \otimes \tau$ is a product, (4.14) implies that the spectral sequence (5.3) collapses at E_3 . Moreover, it can be easily seen that

- (1) $d_2 : \text{Tens}(\mu, \nu) \otimes \zeta \epsilon^k \longrightarrow \text{Ideal}(\sigma\tau) \otimes \epsilon^k$ is never zero,
- (2) $d_2 \equiv 0$ on $\text{Ideal}(\sigma, \tau) \otimes \zeta \epsilon^k \subset E_2^{*, 2k+1}$,
- (3) $d_2 \equiv 0$ on the ϵ^k lines.

The permanent classes

$$(5.5) \quad S = \{\sigma \otimes 1, \tau \otimes 1, \mu \otimes 1, \nu \otimes 1, \sigma \otimes \zeta, \tau \otimes \zeta, 1 \otimes \epsilon\}$$

in E_∞ converge to elements

$$(5.6) \quad T = \{\alpha, \beta u, v, x_1, x_2, e\}$$

in $H^*(A)$, with $\alpha, \beta \in H^{1,2}(A)$, $u, v \in H^{2,2p}(A)$, $x_1, x_2 \in H^{2,6}(A)$ and $e \in H^{2,4p}(A)$. Every element in E_∞ can be expressed as a sum of products from S , for example, $\sigma \mu^i \nu^j \otimes \zeta \epsilon^k = (\sigma \otimes \zeta)(\mu \otimes 1)^i (\nu \otimes 1)^j (1 \otimes \epsilon)^k$. Therefore,

Theorem 5.7. T generates $H^*(A) = H^*(\mathcal{VLC}(3))$ multiplicatively.

We shall look at the relations in $H^*(A)$ in the following section.

5.2. Relations in $H^*(\mathcal{VLC}(3))$. We will use a theorem of Ravenel [16, Appendix 1.4] on Massey products in spectral sequences. Refer to A.2 for the definition of Massey products. The formulation is somewhat complicated, but the main idea, due to Kahn [6], is that we do not want any higher “cross-differentials” to interfere with d_2 . Since $d_r = 0$ for $r \geq 3$, this is not an issue. Thus we can safely state that if $a_i \in E_2^{*,*}$ converge to u_i in $H^*(A)$, then the Massey product $\langle a_1, \dots, a_n \rangle \in E_3$ (when defined) will converge to $\langle u_1, \dots, u_n \rangle$, up to indeterminacy, of course. We shall also make use of the following “juggling theorem”:

Lemma 5.8 ([16, A1.4.6]). *If $\langle u_1, u_2, u_3 \rangle$ and $\langle u_2, u_3, u_4 \rangle$ are defined, then*

$$u_1 \langle u_2, u_3, u_4 \rangle = \langle \bar{u}_1, \bar{u}_2, \bar{u}_3 \rangle u_4,$$

where $\bar{u} = (-1)^{\deg u + 1} u$ and $\deg u$ means total degree of u .

Proposition 5.9. *In $H^*(\mathcal{VLC}(3))$, the multiplicative relations are generated by the following:*

- (1) $\alpha^2 = \alpha\beta = \beta^2 = 0$.
- (2) $\alpha x_2 = -\beta x_1$.
- (3) $\alpha x_1 = \begin{cases} \beta u, & p = 3, \\ 0, & p > 3, \end{cases} \quad \beta x_2 = \begin{cases} -\alpha v, & p = 3, \\ 0, & p > 3. \end{cases}$

$$(4) \quad x_1^2, x_1x_2, x_2^2 = \begin{cases} -ux_2, -uv, vx_1, & p = 3, \\ 0, 0, 0, & p > 3. \end{cases}$$

In addition, we have the following Massey products:

$$\begin{aligned} (5) \quad x_1 &= \langle \alpha, \alpha, \beta \rangle = -1/2 \langle \alpha, \beta, \alpha \rangle = \langle \beta, \alpha, \alpha \rangle. \\ (6) \quad x_2 &= -\langle \alpha, \beta, \beta \rangle = 1/2 \langle \beta, \alpha, \beta \rangle = -\langle \beta, \beta, \alpha \rangle. \\ (7) \quad \langle \alpha \rangle^p &= -\widetilde{\beta\mathcal{P}}^0(\alpha) = u, \quad \langle \beta \rangle^p = -\widetilde{\beta\mathcal{P}}^0(\beta) = v. \end{aligned}$$

We will prove the last three first.

Proof of 5.9(5) and 5.9(6). Using the notation of §A.2, let $(U, \delta) = (E_2^{**}, d_2)$. The elements $a_1 = a_2 = \sigma \otimes 1$, $a_3 = \tau \otimes 1$ are in $E_2^{1,0}$, and have total degree 1. Therefore $\bar{a}_i = a_i$. Now $a_1a_2 = \sigma^2 \otimes 1 = 0$, and $a_2a_3 = \sigma \otimes \tau = d_2(1 \otimes \zeta)$, so we can take $u_1 = 0$, $u_2 = 1 \otimes \zeta$. Then $\langle a_1, a_2, a_3 \rangle = \sigma \otimes 1 \cdot 1 \otimes \zeta + 0 \cdot \tau \otimes 1 = \sigma \otimes \tau$, which converges to x_1 . The indeterminacy is $\alpha H^1 + H^1\beta$, which is 0 since $H^1(A) = \mathbb{F}_p\langle \alpha, \beta \rangle$ and $\alpha^2 = \alpha\beta = \beta^2 = 0$.

$\langle \beta, \alpha, \alpha \rangle = \tau \otimes 1 \cdot 0 + (-1 \otimes \zeta) \cdot \sigma \otimes 1 = \sigma \otimes \zeta$. The minus sign comes from the fact that $\beta\alpha = -\alpha\beta$. It disappears because $1 \otimes \zeta$ and $\sigma \otimes 1$ anticommute.

$$\langle \alpha, \beta, \alpha \rangle = \sigma \otimes 1 \cdot (-1 \otimes \zeta) + 1 \otimes \zeta \cdot \sigma \otimes 1 = -2\sigma \otimes \zeta.$$

Now let $U = C^{**}(A)$, the cobar complex of A . α is represented by $a_1 = a_2 = [a^*]$, β by $a_3 = [b^*]$. Let us calculate $\langle \alpha, \alpha, \beta \rangle$ in U . $u_1 = [a^{2*}]$, $u_2 = [c^*]$, and $\langle a_1, a_2, a_3 \rangle = [a^*|c^*] + [a^{2*}|b^*]$. We now have a representative for x_1 in $C^*(A)$.

Other representatives are

$$(5.10a) \quad x_1 = -\frac{1}{2} \langle \alpha, \beta, \alpha \rangle = \frac{1}{2} ([a^*|c^*] - [c^*|a^*] - [a^*|ba^*]),$$

$$(5.10b) \quad x_1 = \langle \beta, \alpha, \alpha \rangle = [b^*|a^{2*}] - [c^*|a^*] + [ba^*|a^*].$$

The procedure for x_2 is similar. Some representatives are

$$(5.11a) \quad x_2 = -\langle \alpha, \beta, \beta \rangle = -[a^*|b^{2*}] - [c^*|b^*],$$

$$(5.11b) \quad x_2 = -\langle \beta, \beta, \alpha \rangle = -[b^*|ba^*] + [b^*|c^*] - [b^{2*}|a^*],$$

$$(5.11c) \quad x_2 = \frac{1}{2} \langle \beta, \alpha, \beta \rangle = \frac{1}{2} ([b^*|c^*] + [ba^*|b^*] - [c^*|b^*]).$$

□

Proof of 5.9(7). Follows from naturality of the algebraic Steenrod operations.

Proof of 5.9(1). Follows from relations in E_∞ .

$$\text{Proof of 5.9(2). } \alpha x_2 = -\alpha \langle \alpha, \beta, \beta \rangle = -\langle \bar{\alpha}, \bar{\alpha}, \bar{\beta} \rangle \beta = -x_1 \beta = -\beta x_1.$$

Proof of 5.9(3). $\alpha x_1 = \alpha \langle \alpha, \alpha, \beta \rangle = \langle \bar{\alpha}, \bar{\alpha}, \bar{\alpha} \rangle \beta = \langle \alpha, \alpha, \alpha \rangle \beta = u\beta$ for $p = 3$. For larger primes, $\langle \alpha \rangle^3 = 0$, since for $\langle \alpha \rangle^p$ to be defined, we need all lower symmetric Massey products to vanish. Similarly, $\beta x_2 = -\beta \langle \beta, \beta, \alpha \rangle = -\langle \beta, \beta, \beta \rangle \alpha = -v\alpha$ for $p = 3$, while it vanishes for larger primes. In either situation, the products fall to the lower filtration level $F^3 H^3(A) = E_\infty^{3,0}$.

Proof of 5.9(4). Since $(\sigma \otimes \zeta)^2 = -\sigma^2 \otimes \zeta^2 = 0$ in $E_\infty^{2,2} \simeq F^{2/3} H^4(A) = F^2 H^4(A) / F^3 H^4(A)$, we deduce that $x_1^2 \in F^3 H^4(A)$. Since $F^3 H^4(A)$ is spanned by $\{ux_1, ux_2, vx_1, vx_2, u^2, uv, v^2\}$, we can express x_1^2 as a linear combination

$$(5.12) \quad x_1^2 = c_1 ux_1 + c_2 ux_2 + c_3 vx_1 + c_4 vx_2 + c_5 u^2 + c_6 uv + c_7 v^2, \quad c_i \in \mathbb{F}_p.$$

The element x_1^2 has internal degree 12, while the elements on the right-hand side of (5.12) have internal degree larger than or equal to $2p + 6$. For $p > 3$, we are forced to conclude that $x_1^2 = 0$. As usual, the case $p = 3$ requires special attention.

Let $p = 3$. Multiplying both sides of (5.12) by α gives us $\alpha x_1^2 = \beta u x_1$ on the left, and $c_1 \beta u^2 - c_2 \beta u x_1 + c_3 \beta u v - c_4 \beta v x_1 + c_5 \alpha u^2 + c_6 \alpha u v + c_7 \alpha v^2$ on the right. The summands are linearly independent in $H^5(A)$ because their representatives in E_∞ are linearly independent. Therefore, $c_2 = -1$, and all the other c_i are zero. Starting with linear combinations in $H^4(A)$ for $x_1 x_2$ and x_2^2 , then multiplying by α , will yield the other two identities.

These relations all come about from the fact that $\sigma^2 = \tau^2 = \zeta^2 = 0$, $\sigma\tau = -\tau\sigma$ in E_∞ . Since there are no other identities of this type in E_∞ , there are no new relations in $H^*(A)$.

5.3. G of type $G(r, e)$. We next turn our attention to groups G of type

$$G(r, e) = \langle f, g, h \mid f^p = g^{p^{r-2}} = h^p = (g, h) = 1, (f, h^{-1}) = g^{ep^{r-3}}, (f, g) = h \rangle.$$

We will soon see that for our purposes, we can take $e = 1$.

Lemma 5.13. $g^p \in Z(G)$.

Proof. Since $(f, g) = h$, we have $f^{-1}g^{-1}f = hg^{-1}$, and

$$(5.14) \quad (f, g^p) = f^{-1}g^{-p}fg^p = (f^{-1}g^{-1}f)^p g^p = (hg^{-1})^p g^p = h^p g^{-p} g^p = 1.$$

The second-to-last equality follows because $(g, h) = 1$. \square

We will investigate the exact sequence

$$(5.15) \quad \mathbb{Z}/(p)\langle d \rangle \xrightarrow{d \mapsto g^{p^{r-3}}} G(r) \longrightarrow Q(r-1).$$

This is a central extension by Lemma 5.13. As the base is not abelian, we cannot use Theorem 4.15. The cokernel Q has generators $\tilde{f}, \tilde{g}, \tilde{h}$ and relations

$$\tilde{f}^p = \tilde{h}^p = (\tilde{g}, \tilde{h}) = 1, (\tilde{f}, \tilde{h}^{-1}) = \tilde{g}^{ep^{r-3}} = 1, (\tilde{f}, \tilde{g}) = \tilde{h}.$$

Therefore $Q(r-1) \simeq \langle a, b, c \mid a^p = b^{p^{r-3}} = c^p = 1, (a, b) = c \in Z(Q) \rangle$, where $a = \tilde{f}$, $b = \tilde{g}$, $c = \tilde{h}$. For $r = 4$, $Q(r-1) = Q(3) \simeq C(3)$. We will focus on this case presently. The case $r > 4$ is taken up in Section 5.3.2.

5.3.1. Case 1: $r = 4$. For the extension (5.15), we have

$$(5.16) \quad \mathbb{Z}/(p) \xrightarrow{d \mapsto g^p} G(4) \xrightarrow{f, g, h \mapsto a, b, c} Q = C(3)$$

with f, g, h, d in filtration degree 2, 2, 4, $2p$, respectively. The Lie algebra $\mathcal{L}G$ is generated by $\{x = \bar{f}, y_1 = \bar{g}, y_2 = \bar{g}^p, z = \bar{h}\}$. Since $2 + 4 = 2p \Leftrightarrow p = 3$, $[\bar{f}, \bar{h}] = 0$ for $p > 3$. Since $\overline{h^{-1}} = -\bar{h}$, $[\bar{f}, \bar{h}] = -[\bar{f}, \overline{h^{-1}}] = -\bar{g}^{ep} = -e\bar{g}^p$ for $p = 3$. Summarizing, the non-trivial structure of $\mathcal{L}G$ is as follows:

$$\xi(y_1) = y_2, \quad [x, y_1] = z, \quad [x, z] = \begin{cases} -ey_2 & p = 3, \\ 0 & p > 3. \end{cases}$$

For $p = 3$, e can only equal 1, so we see that e does not really matter.

Taking the usual quotient of $\text{Tens}\{x, y_1, y_2, z\}$ by the ideal J of relations as in Definition 2.4,

$$(5.17) \quad \mathcal{V}\mathcal{L}G \simeq \begin{cases} \text{Tens}\{f, g, h\} / \left(\begin{smallmatrix} f^p = g^{p^2} = h^p = 0, fg = h + gf \\ fg^p = g^p f, fh = hf, gh = hg \end{smallmatrix} \right) & p > 3, \\ \text{Tens}\{f, g, h\} / \left(\begin{smallmatrix} f^p = g^{p^2} = h^p = 0, fg = h + gf \\ fg^p = g^p f, fh = -g^p + hf, gh = hg \end{smallmatrix} \right) & p = 3, \end{cases}$$

under the mapping $x \mapsto f$, $y_1 \mapsto g$, $y_2 \mapsto g^p$, $z \mapsto h$.

The central extension of Hopf algebras,

$$(5.18) \quad \mathbb{F}_p[d]/(d^p) \xrightarrow{d \mapsto g^p} \mathcal{V}\mathcal{L}G \xrightarrow{f, g, h \mapsto a, b, c} \mathcal{V}\mathcal{L}Q = \mathcal{V}\mathcal{L}C(3),$$

denoted by $F \xrightarrow{i} A \xrightarrow{\pi} B$, gives us a spectral sequence

$$(5.19) \quad E_2^{*,*}(G(4)) \simeq H^*(B) \otimes (\Lambda[z] \otimes \mathbb{F}_p[w]) \Rightarrow H^*(A),$$

where $z \in E_2^{0,1,2p}$ and $w \in E_2^{0,2,2p^2}$. We shall suppress tensor notation in spectral sequences (e.g. $1 \otimes z = z$) from now on.

Proposition 5.20. *In the spectral sequence (5.19)*

$$d_2(z) = \begin{cases} v - x_1, & p = 3, \\ v, & p > 3. \end{cases}$$

Proof. Put an ordering $h > g > f$ on $A = \mathcal{V}\mathcal{L}G$. A basis for A consists of $\{h^i g^j f^k \mid 0 \leq i, k \leq p-1, 0 \leq j \leq p^2-1\}$. Using the relations (5.17) for A , we see that in the cobar complex $C^*(A)$,

$$(5.21) \quad \delta[g^{p^*}] = \begin{cases} \sum_{i=1}^{p-1} [g^{i^*} | g^{p-i^*}] - [f^* | h^*] - [f^{2^*} | g^*] & p = 3, \\ \sum_{i=1}^{p-1} [g^{i^*} | g^{p-i^*}] & p > 3. \end{cases}$$

Under the edge map $\pi^* : H^*(B) \rightarrow H^*(A)$, $v \mapsto \text{class}(\sum_{i=1}^{p-1} [g^{i^*} | g^{p-i^*}])$ and $x_1 \mapsto \text{class}([f^* | h^*] + [f^{2^*} | g^*])$. The only differential that can cause the right-hand side in (5.21) to be zero is $d_2(z)$. \square

Proposition 5.22. $\widetilde{\beta\mathcal{P}}^0(v) = \widetilde{\beta\mathcal{P}}^0(x_1) = 0$ in $H^3(B)$.

Proof. $v, x_1 \in H^{2,2p}(B)$. The image of $\widetilde{\beta\mathcal{P}}^0 : H^{2,2p}(B) \rightarrow H^{3,2p^2}$ is zero because the internal degrees of the elements of $H^3(B)$ are all less than or equal to $2 + 4p$, which is less than $2p^2$. \square

Proposition 5.23. $d_2(z)$ is not a zero divisor in $H^*(B)$.

Proof. Assume $d_2(z)y = 0$ for some non-zero element $y \in H^{s+t}(B)$ of filtration degree s . Project y to $\bar{y} \in F^{s/s+1}H^{s+t}(B) \simeq E_\infty^{s,t}(C(3))$.

For $p > 3$, the equation $\nu y = 0$ projects to $\nu \bar{y} = 0$. Since ν is not a zero divisor in $E_\infty(C(3))$, $\bar{y} = 0$. This means $y \in F^{s+1}H^*(B)$, which is a contradiction.

For $p = 3$, we have $(v - x_1)y = 0$, or $\nu y = x_1 y$. This projects to $\nu \bar{y} = \sigma \zeta \bar{y}$. Note that $\nu \bar{y} \in E_\infty^{s+2,t}$, while $\sigma \zeta \bar{y} \in E_\infty^{s+1,t+1}$. The two cannot possibly be equal, unless of course they are both zero. But ν is not a zero divisor, and the same contradiction ensues. \square

We conclude

Proposition 5.24. *The spectral sequence (5.19) collapses at E_3 , and*

$$H^*(A) \simeq \begin{cases} H^*(B)/(v - x_1) \otimes \mathbb{F}_p[w], & p = 3, \\ H^*(B)/(v) \otimes \mathbb{F}_p[w], & p > 3. \end{cases}$$

Theorem 5.25. *$H^*(\mathcal{V}\mathcal{L}G(4))$ is generated by $\{\alpha, \beta, u, x_1, x_2, e, w\}$, with $\alpha, \beta \in H^{1,2}$, $x_1, x_2 \in H^{2,6}$, $u \in H^{2,2p}$, $e \in H^{2,4p}$, and $w \in H^{2,2p^2}$. The relations are generated by*

$$\begin{aligned} (1) \quad & \alpha^2 = \alpha\beta = \beta^2 = 0. \\ (2) \quad & \alpha x_2 = -\beta x_1. \\ (3) \quad & \alpha x_1 = \begin{cases} \beta u, & p = 3, \\ 0, & p > 3, \end{cases} \quad \beta x_2 = \begin{cases} -\beta u, & p = 3, \\ 0, & p > 3. \end{cases} \\ (4) \quad & x_1^2, x_1 x_2, x_2^2 = \begin{cases} -u x_2, -u x_1, -u x_2, & p = 3, \\ 0, 0, 0, & p > 3. \end{cases} \end{aligned}$$

In particular, $\mathcal{V}\mathcal{L}G(4)$ is semi-Koszul. The Massey products of Proposition 5.9 still hold, except now $\langle \beta \rangle^p = x_1$ for $p = 3$, and $\langle \beta \rangle^p = 0$, $\langle \beta \rangle^{p^2} = w$ for $p > 3$.

Proof. This is simply a matter of setting $v = 0$ or $v = x_1$ in the relations for $H^*(\mathcal{V}\mathcal{L}C(3))$ (Proposition 5.9). The class w is formally the coboundary of $[g^{p^2*}]$. For $p > 3$, this is equal to the representative of $\langle \beta \rangle^{p^2}$ in the cobar complex. \square

Remark. Notice that $x_2^2 = -u x_2$ implies $x_2 + u$ is a zero divisor, with $x_2(x_2 + u) = 0$. If we had a central extension of 3-groups $\mathbb{Z}/(3) \rightarrow K \rightarrow G(4)$ and the induced spectral sequence (4.12) had $d_2(z) = x_2 + u$, then $x_2 \otimes z \in E_\infty^{2,1}$ would be an indecomposable element of dimension 3. In this case, $\mathcal{V}\mathcal{L}K$ would not be semi-Koszul.

5.3.2. *Case 2: $r > 4$.* Now let $G = G(r, e)$, $r > 4$, $p \geq 3$. The situation is

$$\mathbb{Z}/(p)\langle d \rangle \xrightarrow{d \mapsto g^{p^{r-3}}} G(r) \xrightarrow{f, g, h \mapsto a, b, c} Q(r-1),$$

where

$$Q(r-1) \simeq \langle a, b, c \mid a^p = b^{p^{r-3}} = c^p = 1, [a, b] = c \in \mathcal{Z}(Q) \rangle.$$

The filtration degrees of f, g, h, d are 2, 2, 4, $2p^{r-3}$, respectively.

Let us first deal with finding $H^*(\mathcal{V}\mathcal{L}Q(r-1))$. There is a central extension for Q

$$\mathbb{Z}/(p)\langle z \rangle \xrightarrow{z \mapsto c} Q\langle a, b, c \rangle \xrightarrow{a, b \mapsto x, y} \mathbb{Z}/(p) \times \mathbb{Z}/(p^{r-3})\langle x, y \rangle$$

which is very similar to (5.1). The induced algebra extension

$$\mathbb{F}_p[z]/(z^p) \longrightarrow \mathcal{V}\mathcal{L}Q(r-1) \longrightarrow \mathbb{F}_p[x, y]/(x^p, y^{p^{r-3}}), \quad \deg z = 4,$$

gives us the spectral sequence

$$E_2^{**}(Q(r-1)) \simeq \Lambda[\sigma, \tau] \otimes \mathbb{F}_p[\mu, \nu] \otimes \Lambda[\zeta] \otimes \mathbb{F}_p[\epsilon]$$

with $\sigma, \tau, \mu, \nu, \zeta, \epsilon$ as in (5.3) except ν has internal degree $2p^{r-3}$ and $\nu = \langle \tau \rangle^{p^{r-3}}$. Just as in (5.4), $d_2(\zeta) = \sigma\tau$ and the spectral sequence collapses at E_3 .

Theorem 5.26. $H^*(\mathcal{VL}Q(r-1))$ is generated by $\{\alpha, \beta, x_1, x_2, u, v, e\}$, with $\alpha, \beta \in H^{1,2}$, $x_1, x_2 \in H^{2,6}$, $u \in H^{2,2p}$, $v \in H^{2,2p^{r-3}}$ and $e \in H^{2,4p}$. Relations are generated by

- (1) $\alpha^2 = \alpha\beta = \beta^2 = 0$.
- (2) $\alpha x_2 = -\beta x_1$.
- (3) $\alpha x_1 = \begin{cases} \beta u, & p = 3, \\ 0, & p > 3, \end{cases} \quad \beta x_2 = 0$.
- (4) $x_1^2, x_1 x_2, x_2^2 = \begin{cases} -u x_2, 0, 0, & p = 3, \\ 0, 0, 0, & p > 3. \end{cases}$

The Massey products of Proposition 5.9 still hold, except $\langle \beta \rangle^{p^{r-3}} = v$ for all p .

Proof. The proofs of (1) and (2) are the same as in 5.9. For the second half of (3), $\beta x_2 = -\beta \langle \beta, \beta, \alpha \rangle = -\langle \beta, \beta, \beta \rangle \alpha = 0$ even for $p = 3$. In the proof of (4) for $p = 3$, we are reduced to solving the equation

$$(5.27) \quad x_i x_j = c_1 u x_1 + c_2 u x_2 + c_5 u^2, \quad c_k \in \mathbb{F}_p,$$

by reason of internal dimension. Multiplying both sides by α and equating coefficients yield the desired identities. For example, $\alpha x_2^2 = -\beta x_1 x_2 = 0$, while the right side of (5.27) becomes $c_1 \beta u^2 - c_2 \beta u x_1 + c_5 \alpha u^2$. \square

We now turn to finding $H^*(\mathcal{VL}G(r))$. Notice that since $(f, h^{-1}) = g^{p^{r-3}}$ falls below $\Gamma^6 G$, $[\bar{f}, \bar{h}] = 0$ in $\mathcal{L}G$ even for $p = 3$. Thus (5.17) becomes

$$\mathcal{VL}G(r) \simeq \text{Tens}\{f, g, h\} / \left(\begin{array}{l} f^p = g^{p^{r-2}} = h^p = 0, \quad fg = h + gf \\ fg^p = g^p f, \quad fh = hf, \quad gh = hg \end{array} \right).$$

As in (5.18), setting $\deg d = 2p^{r-3}$ gives us the graded algebra extension

$$\mathbb{F}_p[d]/(d^p) \xrightarrow{i} \mathcal{VL}G(r) \xrightarrow{\pi} \mathcal{VL}Q(r-1)$$

which induces the analogue of (5.19)

$$(5.28) \quad E_2^{*,*}(G(r)) \simeq H^*(\mathcal{VL}Q(r-1)) \otimes \Lambda[z] \otimes \mathbb{F}_p[w].$$

Proposition 5.29. In the spectral sequence (5.28), $d_2(z) = v$.

Proof. In the cobar complex $C^*(\mathcal{VL}G(r))$, $\delta[g^{p^{r-3}*}] = \sum_{i=1}^{p^{r-3}-1} [g^{i*} | g^{p^{r-3}-i*}]$, which is the representative of $\pi^*(v)$. Thus $d_2(z) = v$ for all p . \square

Proceeding in the same way as in Propositions 5.22, 5.23 and 5.24, we obtain

Proposition 5.30. $\widetilde{\beta\mathcal{P}}^0(v) = 0$, and v is not a zero divisor in $H^*(\mathcal{VL}Q(r-1))$. Thus the spectral sequence (5.28) collapses at E_3 and

$$H^*(\mathcal{VL}G(r)) \simeq E_\infty \simeq H^*(\mathcal{VL}Q(r-1))/(v) \otimes \mathbb{F}_p[w].$$

Finally,

Theorem 5.31. $H^*(\mathcal{VL}G(r))$ is generated by $\alpha, \beta, u, x_1, x_2, e, w$. The bidegrees of the generators are as in Theorem 5.25, except $w \in H^{2,2p^{r-2}}$. Relations are as in Theorem 5.25 except now $\beta x_2 = x_1 x_2 = x_2^2 = 0$, $\langle \beta \rangle^{p^{r-2}} = w$ for all p . In particular, $H^*(\mathcal{VL}G(r))$ is semi-Koszul.

5.4. **G metacyclic.** Let G be a metacyclic p -group. That is, there is an extension

$$\mathbb{Z}/(p^m) \longrightarrow G \longrightarrow \mathbb{Z}/(p^n).$$

For p odd, any metacyclic p -group G has presentation

$$G = P(m, n, q, l) = \langle x, y \mid x^{p^m} = 1, y^{p^n} = x^{p^q}, (x, y) = x^{p^l} \rangle$$

for integers $m, n, l, q \geq 1$ satisfying $(p^l + 1)^{p^n} \equiv 1 \pmod{p^m}$ and $(p^l + 1)p^q \equiv p^q \pmod{p^m}$ [4]. We can get all groups up to isomorphism by requiring $l \leq m$, $q \leq m$, $n + l \geq m$ and $q + l \geq m$. Note that $y^{p^{m+n-q}} = 1$.

It is clear that $\mathcal{L}G$ is generated by $x_i = x^{p^{i-1}} \in \Gamma^{2p^{i-1}/2p^{i-1}+1}$, $1 \leq i \leq m$, and $y_j = y^{p^{j-1}} \in \Gamma^{2p^{j-1}/2p^{j-1}+1}$, $1 \leq j \leq n + m - q$, some of which may be zero.

Lemma 5.32. *For all i, j , $[x_i, y_j] = 0$.*

Proof. $y^{-1}xy = x^{p^l+1} \Rightarrow y^{-p^j}x^{p^i}y^{p^j} = x^{p^i(p^l+1)^{p^j}} \Rightarrow (x^{p^i}, y^{p^j}) = x^{p^i((p^l+1)^{p^j}-1)}$. The exponent is $p^i((p^l+1)^{p^j}-1) = p^i(p^{lp^j} + \dots + p^j p^l + 1 - 1) = p^{i+j+l}a$, $a \in \mathbb{Z}$. Therefore, $(x^{p^i}, y^{p^j}) = (x^a)^{p^{i+j+l}} \in \Gamma^{2p^{i+j+l}} \subsetneq \Gamma^{2(p^i+p^j)}$, even for $i = j = 0$. \square

We can break the analysis of $\mathcal{L}G$ down into three cases:

Case 1: $n < q$.

$\xi(y_j) = y_{j+1}$ for $j < n$. Since $y^{p^n} = x^{p^q}$ falls down to a lower filtration level, $\xi(y_n) = \overline{y^{p^n}} = 0$, and there are no y_j for $j > n$. The other generators $\{x_i\}_{1 \leq i \leq m}$ have restriction $\xi(x_i) = x_{i+1}$ for $i < m$ and $\xi(x_m) = 0$. Then

$$\mathcal{V}\mathcal{L}G \simeq \mathbb{F}_p[x, y]/(x^{p^m}, y^{p^n}) \quad \deg x = \deg y = 2.$$

Case 2: $n > q$.

By the same reasoning as above, there are no x_i for $i > q$. $\xi(x_i) = x_{i+1}$ for $i < q$, $\xi(x_q) = 0$. $\xi(y_j) = y_{j+1}$ for $j < n + m - q$, and $\xi(y_{n+m-q}) = 0$. So

$$\mathcal{V}\mathcal{L}G \simeq \mathbb{F}_p[x, y]/(x^{p^q}, y^{p^{n+m-q}}) \quad \deg x = \deg y = 2.$$

Case 3: $n = q$.

In this case, $x^{p^q} = y^{p^n}$ in $\Gamma^{2p^q} = \Gamma^{2p^n}$. Let $z = yx^{p^{m-n}-1}$. Then $(x, z) = x^{-1}x^{1-p^{m-n}}y^{-1}xyx^{p^{m-n}-1} = x^{p^l} = (x, y)$.

Claim. $z^{p^n} = 1$.

Proof. It can be shown by induction that $(yx^a)^b = y^b x^{a[(p^l+1)^b-1]/p^l}$. For $a = p^{m-n} - 1$, $b = p^n$, the exponent of x is

$$(p^{m-n} - 1)((p^l + 1)^{p^n} - 1)/p^l = (p^{m-n} - 1)p^{n+l}c/p^l = (p^m - p^n)c,$$

where $c = 1 + \frac{1}{p^n} \binom{p^n}{2} p^l + \frac{1}{p^n} \binom{p^n}{3} p^{2l} + \dots = 1 + p^l d$, for some $d \in \mathbb{Z}$. Then $z^{p^n} = y^{p^n} x^{(p^m - p^n)c} = x^{p^n(-c+1)} = x^{p^{n+l}d} = 1$ since $n + l \geq m$. \square

So $G \simeq \langle x, z \mid x^{p^m} = 1, z^{p^n} = 1, (x, z) = x^{p^l} \rangle$. Generators for $\mathcal{L}G$ are $\{x_i\}_{1 \leq i \leq m}$ and $\{z_j\}_{1 \leq j \leq n}$, and

$$\mathcal{V}\mathcal{L}G \simeq \mathbb{F}_p[x, z]/(x^{p^m}, z^{p^n}) \quad \deg x = \deg z = 2.$$

Note that the transformation $y \mapsto z$ in G corresponds to the map $y_1 \mapsto z_1 = y_1 - x_1$ in $\mathcal{L}G$.

Theorem 5.33. *Let $G = P(m, n, q, l)$. Then*

$H^*(\mathcal{V}\mathcal{L}G) \simeq \Lambda[x, y] \otimes \mathbb{F}_p[u, v]$, $x, y \in H^{(1,2)}$, $u = \langle x \rangle^a \in H^{(2,2a)}$, $v = \langle y \rangle^b \in H^{(2,2b)}$,
where $a = p^m$, $b = p^n$ if $q \geq n$ and $a = p^q$, $b = p^{m+n-q}$ if $q < n$.

We have finally proven

Theorem 5.34. *Let $p \geq 5$, and let G be a finite p -group of rank 2. Then $\mathcal{V}\mathcal{L}G$ is semi-Koszul.*

APPENDIX A.

A.1. The cobar complex. We summarize the construction of the *cobar complex* of A , which can be used to calculate the cohomology of A . Refer to Priddy [13, Section 1] for more details.

As in Section 2, let A be an algebra with augmentation ideal I . The cobar complex of A , denoted by $C^*(A)$, is given by $C^s(A) = (I^*)^{\otimes s}$. A typical element is written $[\alpha_1 | \cdots | \alpha_s]$ and has bidegree (s, t) , where $t = \sum_{i=1}^s \deg \alpha_i$ is the internal degree inherited from A . Let $C^{s,t}$ be spanned by elements of bidegree (s, t) . There is a differential $\delta : C^{s,t}(A) \rightarrow C^{s+1,t}(A)$, which may be computed as follows. Let μ^* be the composite $A^* \xrightarrow{\mu_A^*} (A \otimes A)^* \xrightarrow{\theta^{-1}} A^* \otimes A^*$, where μ_A^* is dual to the multiplication map μ_A of A , and θ is the isomorphism defined more generally by

$$\theta : M^* \otimes N^* \rightarrow (M \otimes N)^*, \quad \theta(f \otimes g)(m \otimes n) = (-1)^{\deg g \deg m} f(m)g(n)$$

for A -modules M and N . If $\mu^*(\alpha) = \sum_r \alpha'_r \otimes \alpha''_r$, then δ is given by

$$(A.1) \quad \delta([\alpha_1 | \cdots | \alpha_s]) = - \sum_{1 \leq i \leq s; r} (-1)^{e_{i,r}} [\alpha_1 | \cdots | \alpha'_{i,r} | \alpha''_{i,r} | \cdots | \alpha_s],$$

where $e_{i,r}$ is the total degree of $[\alpha_1 | \cdots | \alpha'_{i,r}]$. The cohomology groups of A are then $H^{s,t}(A) = H^s(C^{*,t}(A), \delta)$.

$C^*(A)$ has the structure of a differential graded algebra with product

$$[\alpha_1 | \cdots | \alpha_s] \cup [\beta_1 | \cdots | \beta_{s'}] = [\alpha_1 | \cdots | \alpha_s | \beta_1 | \cdots | \beta_{s'}].$$

The induced product in cohomology coincides with the Yoneda Ext product as given in [1, 2.2].

A.2. Massey products. The following material on Massey products can be found in [16, A1.4] and [8]. We recall the definition of the 3-fold Massey product. Let (U, δ) be a differential bigraded algebra. Let $\alpha_i \in H^{s_i, t_i}(U)$ be represented by a_i , and assume $\alpha_1 \alpha_2 = \alpha_2 \alpha_3 = 0$. Let $s = s_1 + s_2 + s_3$, $t = t_1 + t_2 + t_3$. For any element $a \in U$, let $\bar{a} = (-1)^{\deg(a)+1} a$ where, as usual, degree means total degree. Let $\delta(u_i) = \bar{a}_i a_{i+1}$. Then the Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \in H^{s-1, t}(U)$ is the class represented by the cocycle $\bar{a}_1 u_2 + \bar{u}_1 a_3$. This 3-fold product is not well defined because the choices made in its construction are not unique. The choices of a_i do not matter, but the u_i could each be altered by adding a cocycle $x_i \in U^{s_i + s_{i+1} - 1, t_i + t_{i+1}}$. This would alter $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ by an element of the form $\alpha_1 \llbracket x_2 \rrbracket + \llbracket x_1 \rrbracket \alpha_3 \in \alpha_1 H^{s_2 + s_3 - 1, t_2 + t_3}(U) + H^{s_1 + s_2 - 1, t_1 + t_2}(U) \alpha_3 = \text{Ind} \langle \alpha_1, \alpha_2, \alpha_3 \rangle$. By $\llbracket x \rrbracket$ we mean the class of x in $H^*(U)$. The group $\text{Ind} \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is called the *indeterminacy* of $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$. If the indeterminacy is trivial, then $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is a single class in $H^* * (U)$.

Definition A.2. With notation as above, $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset H^{s-1,t}(U)$ is the coset of $\text{Ind}\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ represented by $\bar{a}_1 u_2 + \bar{u}_1 a_3$.

Similarly, one can define k -fold Massey products $\langle \alpha_1, \dots, \alpha_k \rangle$ for $k > 3$. Let $\alpha_i \in H^{s_i, t_i}$ be represented by a_i , and let $s_{i,j} = \sum_{r=i+1}^j (s_r - 1)$, $t_{i,j} = \sum_{r=i+1}^j t_r$, $s = s_{1,k} + 2$, and $t = t_{1,k}$.

Definition A.3. A collection of cochains $\mathcal{A} = \{a_{i,j} \in U^{s_{i,j}+1, t_{i,j}}\}_{0 \leq i < j \leq k}$ is said to be a *defining system* for the k -fold Massey product $\langle \alpha_1, \dots, \alpha_k \rangle$ if

$$\begin{aligned} a_{i-1,i} &= a_i \text{ for } i = 1, \dots, k, \\ \delta a_{i,j} &= \sum_{i < r < j} \bar{a}_{i,r} a_{r,j}. \end{aligned}$$

The cocycle $c(\mathcal{A}) \in U^{s,t}$ defined by

$$c(\mathcal{A}) = \sum_{r=1}^{k-1} \bar{a}_{1,r} a_{r,k}$$

is called the *related cocycle* of \mathcal{A} . The k -fold Massey product $\langle \alpha_1, \dots, \alpha_k \rangle \subset H^{s,t}(U)$ consists of all classes $\llbracket c(\mathcal{A}) \rrbracket$, where \mathcal{A} ranges over all defining systems of $\langle \alpha_1, \dots, \alpha_k \rangle$.

It can be shown that the set $\langle \alpha_1, \dots, \alpha_k \rangle$ does not depend on the choice of representatives of the α_i . The k -fold product is then a cohomology operation of k variables that is defined if all the lower products $\langle \alpha_i, \dots, \alpha_j \rangle$ for $0 < j - i < k - 1$ are defined and contain zero. Here the double product $\langle \alpha_i, \alpha_{i+1} \rangle$ is understood to be the regular product $\alpha_i \alpha_{i+1}$. The indeterminacy can get quite complicated, however [16, A1.4.4]. Kraines has constructed a “restricted” k -fold Massey product on the cohomology of topological spaces which has no indeterminacy [8, Section 3]. This construction can be carried over to the Hopf algebra setting [7, Section 3]. If $\alpha_i = \alpha$ for all i and $\alpha^2 = 0$, one can define the *symmetric* Massey product $\langle \alpha \rangle^k$.

Definition A.4. Let $\alpha \in H^{s,t}(U)$ be a class such that $\alpha^2 = 0$. A collection of cocycles $\mathcal{A} = \{a_i \in U^{i(s-1)+1, it}\}_{i=1}^{k-1}$ is called a *defining system* for the symmetric Massey product $\langle \alpha \rangle^k$ if

$$\begin{aligned} a_1 &\text{ represents } \alpha, \\ \delta a_i &= \sum_{r=1}^{i-1} \bar{a}_r a_{i-r} \text{ for } 2 \leq i \leq k-1. \end{aligned}$$

The cocycle $c(\mathcal{A}) \in U^{k(s-1)+2, kt}$ defined by

$$c(\mathcal{A}) = \sum_{r=1}^{k-1} \bar{a}_r a_{k-r} \in U^{k(s-1)+2, kt}$$

is called the *related cocycle* of \mathcal{A} . The symmetric product $\langle \alpha \rangle^k$ is then the set of all classes $u \in H^{k(s-1)+2, kt}(U)$ which are represented by some $c(\mathcal{A})$, where \mathcal{A} is a defining system for $\langle \alpha \rangle^k$. If $\langle \alpha \rangle^k$ is defined and equal to a single homology class then $\langle \alpha \rangle^k$ is said to be defined with *zero indeterminacy*.

Lemma A.5. *Let p be odd, and let A be a Hopf algebra over \mathbb{F}_p .*

- (1) *If $\alpha \in H^{2s+1,t}(A)$, then $\langle \alpha \rangle^p$ is defined with zero indeterminacy [7, Lemma 14].*
- (2) *If $\alpha \in H^{1,t}(A)$, then $\langle \alpha \rangle^p = -\widetilde{\beta\mathcal{P}}^0(x)$ [11, Remark 11.11].*

A.3. Calculations. Let

$$(A.6) \quad \begin{aligned} A &= \mathbb{F}_p[x]/(x^{p^n}), & \deg x = m \text{ even}, n \geq 1, \quad p \text{ odd}, \\ A &= \mathbb{F}_2[x]/(x^{2^n}), & \deg x = m, n > 1, \quad p = 2. \end{aligned}$$

The augmentation in both cases is defined by $\epsilon(x) = 0$. It is known that the cohomology of such a truncated polynomial algebra is the tensor product of an exterior algebra on a one-dimensional class and a polynomial algebra on a two dimensional class. We would like to identify specific generators in the cobar complex $C^*(A) = \text{Tens}(I(A)^*)$.

Choose for $I(A)$ the natural basis $\{x^i, 0 < i < p^n\}$. Let $\{y_i\}$ be the dual basis for I^* . Then

$$(A.7) \quad \delta[y_k] = \sum_{\substack{i+j=k \\ 0 < i < k}} [y_i|y_j].$$

Clearly, in $C^{1,2m}(A)$, $z = [y_1]$ is a cocycle. Let $e = \sum_{i=1}^{p^n-1} [y_i|y_{p^n-i}]$. A routine calculation shows that e is a nonbounding cocycle. By abuse of notation, we can thus take z and e to be the generators of $H^*(A)$.

Lemma A.8. *Let A be a truncated polynomial algebra as in (A.6). Then $H^*(A) \simeq \Lambda(z) \otimes \mathbb{F}_p[e]$, with $z \in H^{1,m}$, $e \in H^{2,mp^n}$ as defined above. Moreover, the symmetric Massey product $\langle z \rangle^{p^n} = e$ with zero indeterminacy.*

Proof. Choosing $a_i = [y_i]$ gives a defining system for $\langle z \rangle^{p^n}$. Its related cocycle is $\sum_{r=1}^{p^n-1} [y_r|y_{p^n-r}]$, which represents e . There is no indeterminacy because for each i , there is no other choice of a_i . \square

Remark. If $p = 2$, $n = 1$, then $A = \mathbb{F}_2[x]/(x^2) \simeq \Lambda[x]$, and $H^*(A) \simeq \mathbb{F}_2[z]$ with $z = [x^*] \in H^{1,\deg x}$.

Summarizing,

Theorem A.9. *Let $A = \mathbb{F}_p[x]/(x^{p^n})$, then the following Steenrod operations and symmetric Massey products are present in $H^*(A)$:*

- (1) $\langle z \rangle^{p^n} = e$, $\widetilde{\mathcal{P}}^1 e = e^p$ for p odd,
- (2) $\widetilde{\beta\mathcal{P}}^0 z = -e$ for p odd, $n = 1$,
- (3) $\langle z \rangle^{2^n} = e$, $\widetilde{Sq}^2(e) = e^2$ for $p = 2$, $n \geq 2$,
- (4) $\widetilde{Sq}^1(z) = z^2$ for $p = 2$, $n = 1$.

Proof. These are consequences of the axioms in Theorem 2.2 and of the results just stated. \square

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